Linear Algebra
May 12, 2016 9:10 AM

Section 1: Systems of linear Equations

Def: Linear equation

A linear equation is a sum of terms that are all linear in the unknown variables:

$$\text{ex: } -1x+2y+3z=10$$
$$-2t+4t_2+5t_3=100$$

Def: System of linear equations.

A system of linear equations is one or more lineal equations in the same variables

$$\text{eg. } \left\{ \begin{array}{l} 1x+2y+3z=0 \\ -x+3y-4z=16 \end{array} \right\}$$

A system of linear equations is called 'consistent' if it has one or more solutions and 'inconsistent' if it has no solutions.

We can represent linear equations with matrices:

(*) The coefficient matrix:

(1) order variables consistently across equations
(2) fill matrix, row by row with the coefficients of the equation:

eg. $x+y=0$, $-z+x-y=10$

$$(1)\quad \begin{array}{l} 1x+1y+0z=0 \\ 1x-1y-1z=0 \end{array} \longrightarrow (2) \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

(*) The Augmented Matrix

(1) Construct the coeffient matrix
(2) Add column of constants to the end.

eg. $x+y=0$, $-z+x-y=10$

① $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \longrightarrow$ ② $\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 10 \end{array}\right]$

The Augmented Matrix holds all of the information of the Linear system so we can use it to solve for the solutions.

We define three 'elementary' operations that that leave the system unchanged, but help us solve for the solutions:

1) Interchange two rows

eg: $\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 10 \end{array}\right]$

$$\downarrow$$

$$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 10 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

2) Multiply an equation by a non-zero number

eg: $\left[\begin{array}{ccc:c} 1 & -1 & -1 & 10 \\ 1 & 1 & 0 & 0 \end{array}\right]$

$$\downarrow$$

$$\left[\begin{array}{ccc:c} 2 & -2 & -2 & 20 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

3) Add a multiple of an equation to another equation.

eg: $\left[\begin{array}{ccc:c} 2 & -2 & -2 & 20 \\ 1 & 1 & 0 & 0 \end{array}\right]$

$$\downarrow$$

$$\left[\begin{array}{ccc:c} 2 & -2 & -2 & 20 \\ 0 & 2 & 1 & 10 \end{array}\right]$$

Gauss and Gauss-Jordon Elimination

An algorithm exists that can solve <u>any</u> linear system of equations by applying elementary operations to the system's

Augmented matrix.

Gaussian Elimination:

(1) If matrix is all zeros stop

(2) Find the first non-zero column and move the row with the non-zero entry to the top.

(3) Normalize the top row by dividing by the first entry

(4) Make all entries below the leading one zero by subtracting the appropriate multiple of the top row

(5) Repeat (1–5) on the rows below the leading one.

How does this lead to a solution to any equation?

Example: system with zero solutions:

$$\begin{aligned} x + y + z &= 10 \\ x + y &= 1 \\ x + y &= 2 \end{aligned} \longmapsto \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{array}\right]$$

(1) Perform Gaussian Elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 0 & -1 & -9 \\ 0 & 0 & -1 & -8 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

(2) Translate back to linear equations:

$$x + y + z = 10, \quad z = 9, \quad \underline{\underline{0 = -1}}$$

$$\Rightarrow 0 \neq -1 \Rightarrow \underline{\text{No solution}}.$$

Example: system with 1 solution

$$\begin{aligned} 1x + 1y - 3z &= 3 \\ -2x - 1y &= -4 \\ 4x + 2y + 3z &= 7 \end{aligned} \longmapsto \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ -2 & -1 & 0 & -4 \\ 4 & 2 & 3 & 7 \end{array}\right]$$

(1) Perform Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \end{array}\right. \quad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \end{array}\right. \quad \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \end{array}\right.$$

① Perform Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ -2 & -1 & 0 & -4 \\ 4 & 2 & 3 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 1 & -6 & 2 \\ 0 & -2 & 15 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -6 & 2 \\ 0 & 0 & 3 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -6 & 2 \\ 0 & 0 & 1 & -1/3 \end{array}\right]$$

② Translate back to linear equations:

$$\boxed{z = -1/3}, \quad y - 6z = 2, \quad x + y - 3z = 3$$

$$y + 2 = 2 \Rightarrow \boxed{y = 0}$$

$$x + 0 - 3\left(\frac{-1}{3}\right) = 3 \Rightarrow \boxed{x = 4}$$

Example : System with infinitely many solutions

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 \quad\quad &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned} \quad \mapsto \quad \left[\begin{array}{cccc|c} 1 & -2 & -1 & +3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right]$$

① Perform Gaussian Elimination

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

② Translate back to equations:

$$x_1 - 2x_2 + x_4 = 2$$

$$x_3 - 2x_4 = 1$$

③ Assign parameters to each variable <u>without</u> a corresponding leading one.

$$\underline{\underline{x_2 = t, \quad x_4 = s.}}$$

$$x_1 - 2s + t = 2 \Rightarrow x_1 = 2 + 2s - t$$

$$x_3 - 2t = 1 \Rightarrow x_3 = 1 + 2t$$

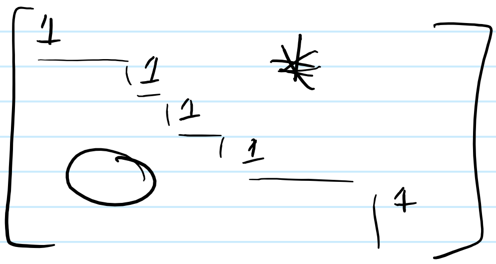
$$x_3 - 2t = 1 \Rightarrow \boxed{x_3 = 1+2t}$$

any $t, s \in \mathbb{R}$ is a solution!

Gauss-Jordon Elimination

After performing Gauss Elimination, a matrix is said to be in 'row-echelon' form. These look like staircases of leading ones

$M_{re} =$



The Gauss-Jordon Elimination goes one step further in finding a solution by putting the Augmented matrix in an even more complete form:

Gauss-Jordon Elimination

Same as Gauss Elimination except after clearing column below leading one, clear the column of entries above the leading one in the same fashion.

$\longrightarrow$ result is in 'reduced-row-echelon' form.

Def: Rank of a Matrix

Rank of a Matrix is # of leading ones when Matrix is in echelon form.

Def: Homogeneous System

All constants are zero!

Solving Homogenous Systems

All homogeneous solutions have the a solution where all

Variables are equal to zero called the trivial solution
In addition we want to parametrize other solutions called, Non-trivial solutions. This works very similarly to other systems with a caveat we'll see through example:

* Note: if the number of variables is greater than the number of equations then there are <u>no</u> nontrivial solutions

<u>Example</u>!

$$\begin{aligned} x_1 - 2x_2 + x_3 + x_4 &= 0 \\ -x_1 + 2x_2 \qquad + x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 \qquad &= 0 \end{aligned} \longmapsto \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 0 \end{array}\right]$$

① <u>Gauss-Jordan</u>

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

② Back to equations: assign $x_2 = s$, $x_4 = t$

$$x_1 - 2s - t = 0 \Rightarrow x_1 = 2s + t$$
$$x_3 + 2t = 0 \Rightarrow x_3 = -2t$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s+t \\ s \\ -2t \\ t \end{bmatrix}$$ <u>General solution</u>

③ Split s, and t contribution to <u>General Solution</u>.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Here we call $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ <u>specific</u> solutions

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow X = sX_1 + tX_2$$

All "basic" solutions are linear combinations of these specific solutions.

Matrix Multiplication

Arbitrary matrices can be multiplied together!

Lets re-examine linear systems:

We could represent the following system equations in a new way:

ex:
$$\begin{aligned} x + y - z &= 1 \\ -x + z &= 2 \end{aligned} \mapsto \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

What do we mean by this? Well if we use this rule, we recover our equation:

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1x + 1y - 1z \\ -x + 0y + 1z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

so each row represents an equation

We can extend this method to more colums and more columns of output

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \end{bmatrix}$$

so any $n \times m$ matrix can be multiplied by an $p \times l$ matrix

$$\boxed{\begin{array}{ccccc} A & * & B & = & C \\ (n \times m) & & (p \times l) & & (n \times l) \end{array}}$$

if $\boxed{m = p}$

Properties

(1) $IA = A$, $BI = B$ * Identity

(1) $\mathbb{1}A = A$, $B\mathbb{1} = B$ * Identity

(2) $A(BC) = (AB)C$ * Associative

(3) $A(B+C) = AB + AC$ * Distributive (on the right)

(4) $(A+B)C = AC + BC$ * Distributive (on the left)

(5) $cAB = (cA)B = A(cB)$ * Commutative scalars

(6) $(AB)^T = B^T A^T$

<u>Example</u>: if $AA - BB = (A-B)(A+B)$ show that $[A,B] = 0$

$$(A-B)(A+B) = A^2 - BA + AB - B^2$$

$$\Rightarrow (A-B)(A+B) = A^2 - B^2 \quad \text{if } AB - BA = 0$$

$$\Rightarrow \boxed{AB = BA}$$

<u>Def.</u> Symmetric Matrices

if $A = A^T \Rightarrow A$ is a symmetric matrix.

<u>Example</u>: if A, B as symmetric and $AB = BA$ then AB is also symmetric.

$$(AB)^T = B^T A^T = BA = AB$$

$$\Rightarrow (AB) = (AB)^T \Rightarrow AB \text{ is symmetric.}$$

The Matrix Inverse

Def: A matrix C is called the inverse of a matrix A, is called the inverse of A if:

$$\boxed{AC = \mathbb{1} \quad \text{and} \quad CA = \mathbb{1}}$$

Denoted A^{-1}

<u>Note</u>: only square matrices have inverses.

Inversion Algorithm

Gauss - Jordan on $[A \mid I]$

Example: What is A^{-1}

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

① $$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & -3/2 & 11/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -3/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & -1/2 \end{array}\right] = [I \mid A^{-1}]$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ -1 & 1 & -1 \end{bmatrix}$$

Note: If this process fails it implies that A is not invertible.

Properties

(1) $\mathbb{1}^{-1} = \mathbb{1}$

(2) if A invertible, A^{-1} invertible and $(A^{-1})^{-1} = A$

(3) if A and B invertible, $(AB)^{-1} = B^{-1}A^{-1}$

(4) if A_i, $i \in 1 \cdots k$ invertible $(A_1 \cdots A_k)^{-1} = A_k^{-1} \cdots A_1^{-1}$

(5) if A invert. $(A^T)^{-1} = (A^{-1})^T$

(6) if A invert $c \neq 0$ $(cA)^{-1} = \frac{1}{c}A^{-1}$

(7) if A invert. $(A^k)^{-1} = (A^{-1})^k$.

Example: simplify $C^T B (AB)^{-1} [C^{-1} A^T]^T$ for A, B, C invertible

Example: simplify $C^T B(AB)^{-1}[C^{-1}A^T]^T$ for A, B, C invertible

$$C^T B(AB)^{-1}[C^{-1}A^T]^T = C^T \underbrace{BB^{-1}}_{\mathbb{1}} A^{-1}[C^{-1}A^T]^T$$

$$= C^T A^{-1}[A(C^{-1})^T]$$

$$= C^T \mathbb{1}(C^T)^{-1}$$

$$= \underline{\mathbb{1}} \checkmark$$

Triangular Matrices

→ invertible iff all diagonal elements are non-zero.

Conditions for invertibility

All are equivalent:

(1) A is an invertible matrix
(2) $AX = 0$ only has a trivial solution.
(3) $AX = B$ has a solution for all $\underline{B}$.
(4) $\exists\, C$ s.t. $AC = \mathbb{1}$, $CA = \mathbb{1}$.

The Determinant

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Section #2 – The Determinant Function

The determinant ... is a number that characterizes a matrix. For a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc$$

What about a general N×N matrices?

Def: Cofactor C_{ij}

The (i,j)th cofactor $(C_{ij}(A))$ of a matrix A is Defined as $\boxed{C_{ij}(A) = (-1)^{i+j}\det(A_{ij})}$ where A_{ij} is the matrix obtand by deleting the ith row and jth column of A.

Example: $A = \begin{bmatrix} 2 & -3 & 7 \\ 5 & -1 & 6 \\ 11 & 0 & 8 \end{bmatrix}$

What is $C_{33}(A)$?

$$C_{33}(A) = (-1)^6 \det\begin{pmatrix} 2 & -3 \\ 5 & -1 \end{pmatrix}$$

$$= -2 + 15$$

$$= 13$$

The Cofactor Expansion Theorem

If A is a square matrix, the determinant $\det(A)$ is equal to the Laplace expansion along any row or column of A.

eg. $\det(A) = a_{11}C_{11}(A) + a_{12}C_{12}(A) + a_{1}\dots \quad a_{1n}C_{1n}(A)$

Example: $A = \begin{bmatrix} 5 & -1 & 7 \\ 9 & -3 & 6 \\ 4 & 8 & 0 \end{bmatrix}$ $\det(A) = ?$

$$\det(A) = 5\left((-1)^{1+1}\det\begin{pmatrix} -3 & 6 \\ 8 & 0 \end{pmatrix}\right) - 1\left((-1)^{1+2}\det\begin{pmatrix} 9 & 6 \\ 4 & 0 \end{pmatrix}\right) + 7\left((-1)^{1+3}\det\begin{pmatrix} 9 & -3 \\ 4 & 8 \end{pmatrix}\right)$$

$$\det(A) = 5\left((-1)^{1+1}\det\begin{pmatrix}-3 & 6\\ 8 & 0\end{pmatrix}\right) - 1\left((-1)^{1+2}\det\begin{pmatrix}9 & 6\\ 4 & 0\end{pmatrix}\right) + 7\left((-1)^{1+3}\det\begin{pmatrix}9 & -3\\ 4 & 8\end{pmatrix}\right)$$

$$= 5(-48) + 24 + 7(72+12)$$

$$= 324$$

Example $A = \begin{bmatrix} 2 & -1 & 6 & 3\\ 1 & 0 & 8 & 7\\ 7 & 9 & 0 & 2\\ 4 & 0 & 0 & 8\end{bmatrix}$ $\det A = ?$

Choose row 4

$$\det A = 4\left(-1\det\begin{pmatrix}-1 & 0 & 3\\ 0 & 5 & 7\\ 9 & 0 & 8\end{pmatrix}\right) + 8\left(1\det\begin{pmatrix}2 & -1 & 6\\ 1 & 0 & 5\\ 7 & 9 & 0\end{pmatrix}\right)$$

$$= -4\big(4(-8-27)\big) + 8\big(-5(0)\big)$$

$$= +16(35)$$

$$= \underline{420}\ !$$

Elementary Ops and The Determinant

* A + swap row/column $\rightarrow B \Rightarrow \det B = -\det A$
* A + scale row by $k \rightarrow B \Rightarrow \det B = k \det A$
* A + add mult. of row $\rightarrow B \Rightarrow \det B = \det A$
* $\det(kA) = k^n \det(A)$
* $\det(A^T) = \det(A)$
* $\det(A^{-1}) = (\det(A))^{-1}$
* $\det(A^k) = (\det A)^k$
* $\det(A_1 \dots A_N) = \det(A_1) \dots \det(A_N)$

Example : $\det A = 2$ and $\det B = -3$

$$\det(A^3 B^{-1} A^T B^2)$$

$$\det(A^3 B^{-1} A^T B^2) = (\det(A))^3 (\det(B))^{-1} \det(A) \det(B)^2$$

$$= \frac{2^4}{-3}\, 9 = -3 \times 16$$

$$\boxed{= -48}$$

Def: The adjoint adj A

The adjoint $\operatorname{adj} A = \left[C_{ij}(A)\right]^T$

The Adjoint Formula:

$$\boxed{A \operatorname{adj}(A) = \det(A)\, \mathbb{1}}$$

$$\Rightarrow \text{ if } \det(A) \neq 0 \quad \boxed{A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}}$$

Def: The trace

$$\operatorname{tr}(A) = \sum_i a_{ii}$$

Cramers Rule

Denote $A_i(B)$ as the matrix A replacing column i by the column matrix B

Consider linear system $AX = B$

$$\text{if } X = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}, \quad x_i = \frac{\det(A_i(B))}{\det(A)}$$

The Determinant by Row Reduction:

Example:

Vectors

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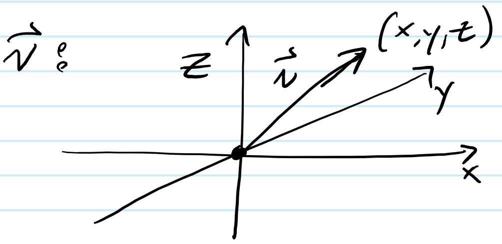
Matrix Form of a Vector

$\vec{v}$:

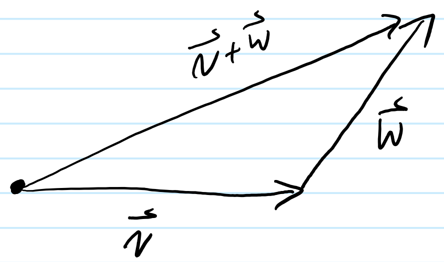


$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the matrix form of $\vec{v}$.

Note: here $\vec{v}$ is represented by a column matrix but $\exists$ formulations where v is a row vector that are equivalent.

Adding Vectors

Two vectors can be added by adding each component together. Visually, this looks like arranging the vectors tip to tail and the resulting vector connects the first tail to the last tip

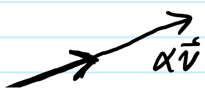


if $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, we add component by component

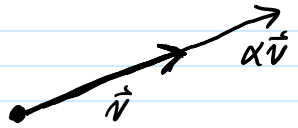
$$\boxed{\vec{v} + \vec{w} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}}$$

Scaling Vectors

Vectors can be scaled (change length but not direction)



In matrix form:



In matrix form:

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \alpha\vec{v} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha z \end{bmatrix}$$

Properties of Vectors

(1) $\vec{v} = \vec{w}$ if they are equal as matrices

(2) $\|\vec{v}\| = \sqrt{x^2+y^2+z^2}$

(3) $\vec{v} = \vec{0}$ if and only if $\|\vec{v}\| = 0$

(4) $\|\alpha\vec{v}\| = |\alpha|\,\|\vec{v}\|$ for any scalar α.

The Dot Product

if $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ then the dot product $\bullet$ is defined

to be : $\boxed{\vec{v}\cdot\vec{w} = x_1x_2 + y_1y_2 + z_1z_2}$

Properties :

(1) $\vec{v}\cdot\vec{w}$ is a number

(2) $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$

(3) $\vec{v}\cdot\vec{0} = 0 = \vec{0}\cdot\vec{v}$

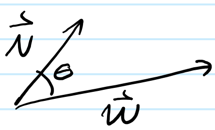
(4) $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$

(5) $\alpha(\vec{v}\cdot\vec{w}) = (\alpha\vec{v})\cdot\vec{w} = \vec{v}\cdot(\alpha\vec{w})$

(6) $\vec{v}\cdot(\vec{w}+\vec{u}) = \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{u}$

(7) $\vec{v}\cdot(\vec{u}-\vec{w}) = \vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{w}$

Angle θ, between Vectors



The angle between two vectors can be defined through the dot product:

$$\boxed{\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}}$$

if $\vec{v}\cdot\vec{w} > 0 \Rightarrow \theta$ is acute

if $\vec{v}\cdot\vec{w} = 0 \Rightarrow \theta$ is a right angle

if $\vec{v}\cdot\vec{w} < 0 \Rightarrow \theta$ is obtuse.

Projections

Let $\vec{v}$ and $\vec{d} \neq 0$ be vectors.

The projection of $\vec{v}$ on $\vec{d}$ is defined as

$$\boxed{proj_{\vec{d}}(\vec{v}) = \frac{\vec{v}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d}}$$

* Properties : $\vec{d}\cdot\left(\vec{v} - proj_{\vec{d}}(\vec{v})\right) = 0$

eg. $\vec{d} \perp (\vec{v} - proj_{\vec{d}}(\vec{v}))$

Any vector $\vec{v}$ can be decomposed into components parallel and perpendicular to a vector $\vec{d}$:

$\vec{v} = \vec{v}_{\parallel} - \vec{v}_{\perp}$ * $\vec{v}_{\parallel} = proj_{\vec{d}}(\vec{v})$

* $\vec{v}_{\perp} = \vec{v} - proj_{\vec{d}}(\vec{v})$

Definition of a Line

We can define a line using a parameter 't'

$\vec{\ell}(t) = \vec{\ell}_0 + t\vec{d}$ Vector Equation   Scalar Equations

-or- $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\boxed{\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}}$$

The vector $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the 'direction' of the line.

Example $\vec{\ell}_1 = \begin{aligned} x &= 2 - t \\ y &= 5 + 2t \\ z &= -1 + 3t \end{aligned}$ $\vec{\ell}_2 = \begin{aligned} x &= 3 + 2s \\ y &= 2 - 3s \\ z &= -6 - 4s \end{aligned}$

do $\vec{\ell}_1$ and $\vec{\ell}_2$ intersect?

$$\begin{aligned} 2 - t &= x_{int} = 3 + 2s \\ 5 + 2t &= y_{int} = 2 - 3s \\ -1 + 3t &= z_{int} = -6 - 4s \end{aligned} \implies \begin{aligned} -t - 2s &= 1 \\ 2t + 3s &= -3 \\ 3t + 4s &= -5 \end{aligned}$$

$$\begin{bmatrix} -1 & -2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -9 \\ 0 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & -3 \\ 3 & 7 & -5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -5 \\ 0 & -2 & -8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -9 \\ 0 & 1 & 5 \\ 0 & 1 & 4 \end{array}\right]$$

Planes

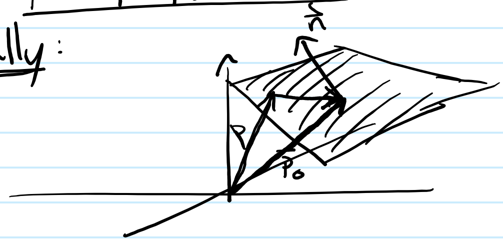
Planes are geometric objects that can be described by the points spanned by two vectors. There are two canonical forms for defining a plane:

Point & Normal (aka Vector Equation of a Plane)

A plane π can be defined by a point $P_0 \in \pi$ and the vector normal to the surface $\vec{n}$. A vector $\vec{p}$ is contained in π iff.

$$\boxed{\vec{n} \cdot (p - p_0) = 0}$$

Visually:



General Form (aka scalar form)

For $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\vec{p}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ then we evaluate the vector form of the plane to find:

$$(x, y, z) \in \pi \text{ iff: } \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

The Cross Product

The cross product is defined as $\vec{v} \times \vec{w}$:

if $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

Properties:

(1) $\vec{v} \times \vec{w}$ is orthogonal to $\vec{w}$ and $\vec{v}$

$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix}$ $\begin{bmatrix} y_2 \\ z_2 \end{bmatrix}$

$$\vec{v}\times\vec{w} = \begin{bmatrix} \det\begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix} \\ -\det\begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix} \\ \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \end{bmatrix}$$

or $\vec{v}\times\vec{w} = \det\begin{bmatrix} i & x_1 & x_2 \\ j & y_1 & y_2 \\ k & z_1 & z_2 \end{bmatrix}$

(1) $\vec{v}\times\vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$

(2) $\vec{v}\times\vec{w} = 0$ iff $\vec{v}$ and $\vec{w}$ are //

More Properties

(1) $\vec{v}\times\vec{w}$ is a vector

(2) $\vec{v}\times 0 = 0 = \vec{0}\times\vec{v}$

(3) $\vec{v}\times\vec{w} = -\vec{w}\times\vec{v}$

(4) Linearity.

The Triple product

The parallel piped spanned by $\vec{v}, \vec{u}, \vec{w}$ has volume V of:

$$\boxed{V = |\vec{u}\cdot(\vec{v}\times\vec{w})| = |\det[\vec{u}\ \vec{v}\ \vec{w}]|}$$

$\vec{u}\cdot(\vec{v}\times\vec{u})$ is called the triple product!

Examples For Linear Equations

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1.2

#1) Solve the following linear systems:

a) $2x - 3y = 4$, $x - 3y = 1$ $\longmapsto$ $x = 3$, $y = 2/3$

b) $2x + 2y - 3z = 1$, $x + z = 5$, $3x + 4y - 7z = -3$ $\longmapsto$ $x = 5 - t$, $y = \frac{5}{2}t - \frac{9}{2}$, $z = t$

c) $4x + y - 8z = 1$, $3x - 2y + 3z = 5$, $-x + 8y - 25z = -3$ $\longmapsto$ $\nexists$ a solution

#2 How many solutions? (use Rank theorem)

Find the rank of each system of #1

List number of variables and connect with number of solutions.

1.3 Homogeneous Systems

Express the general solution as a linear comb. of basic solutions

a) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 0 \\ 1 & 8 & 3 \end{bmatrix}$ $\Big\downarrow$

b) $\begin{bmatrix} 1 & -2 & 0 & 3 \\ -3 & 6 & 1 & -4 \\ 2 & 4 & 1 & 1 \end{bmatrix}$ $\Big\downarrow$

No basic Solution

$X = 0$

$X_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \end{bmatrix}$

1.4 Matrix Multiplication

#1) True or False

a) If AB defined & square A, B BA defined? $\longrightarrow$ True

b) if A^2 defined, A is square $\rightarrow$ True

c) if A has a column of zeros, AB does too $\rightarrow$ False

#2) Simplify

a) $A(3B - C) + A^3C$

b) $(A-B)(A-B) - A^2 + B^2$

1.5 Inverses

Find a Matrix A s.t.

a) $(5A)^T = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}^{-1} \longrightarrow A = \frac{1}{25}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

b) $(A^{-1} - 3\mathbb{1})^T = 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow A = \frac{1}{34}\begin{bmatrix} 23 & -15 \\ -10 & 8 \end{bmatrix}$

Examples For Determinants

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Compute these Determinants

$$\begin{bmatrix} 6 & -9 \\ -11 & 21 \end{bmatrix} \longrightarrow 0$$

$$\begin{bmatrix} a+1 & a \\ a & a-1 \end{bmatrix} \longrightarrow -1$$

$$\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \longrightarrow 2abc$$

~~Komptude~~

$$\det\left[A^3 B^{-1} C^T B^2 A^{-1}\right]$$

if

$\det A = -2$

$\det B = 3$

$\det C = -1$