Electricity and Magnetism

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1 The Lorentz Force

In your study of mechanics you learned that the motion of all particles is determined by Newtons Laws. Specifically, if we know the forces to which a particles is subject to we can, in principle know the where the particle will travel in the future.

$$\sum \vec{F} = ma = m\frac{d^2x}{dt^2} = \frac{dp}{dt} \tag{1}$$

Of course, in practice the details of the force can make it difficult to solve for the trajectory, the important idea is that all the information needed is there. There is nothing else that determines the motion of particles, or anything for that matter.

The importance of this statement can be easily underestimated. While it may seem that the motion of things is a fraction of the rich diversity of experiences that make up our daily lives, the truth is that much of our experience is shaped the motions of microscopic particles. Features as diverse as the colour of objects around us to our ability to sense, feel and think, to the thick, viscous flow of molassas can all be discribed by the details of microscopic motion.

To that end, this course will focus on one particular force called the *Lorentz Force* or Electormagnetic Force

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{2}$$

Admittedly, the Lorentz force may seem a lot more complex than forces that you've studied previously so lets step through the equation variable by variable. Firstly, the force on a particle depends on the charge, q, of the particle. The charge is a property of a particle of interest and can be measured. The charge has units of Coulombs (C). The force also depends on three vectors, the velocity, $\vec{v}$, the electric field, $\vec{E}$, and the magnetic field $\vec{B}$

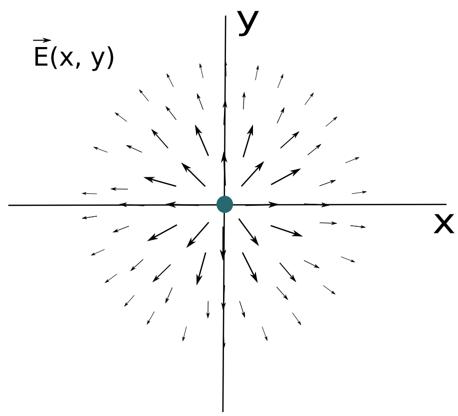
In principle, if you were given a the charge, electric and magnetic field, then you'd be left with the (somewhat tricky) task of solving Newtons equations for the behaviour of your particle. But, what are the electric and magnetic field?

As we can see from the Lorentz force is subject to the state of the electric and magnetic field, but it is very rare in physics for a object to be subject to the state of some quantity and while the reverse is not also true. Newtons third law is a perfect example of this symmetry in physics, but there are more cases of this reciprocity and so it is that just as the motion and velocity of charged particles are subject to the state of the electric and magnetic fields so are the electric and magnetic fields determined by the positions and velocities of charged particles.

This is the study of electricity and magnetism. The study of charged particles, the fields the produce and the results of there interactions.

2 The Electric Field

The electric field is called a *field* because it is defined everywhere in space. The idea is that a charged particle is subject to the value of the electric field at its exact position so, for this effect to be possible for any charged particle, anywhere in the universe, there must be an electric field vector defined for every point in space.



This type of force, that depends on position, is nothing new. Remember that when you studied simple harmonic motion the force of the spring depended on the position of the mass.

$$\vec{F} = -k\vec{x} \tag{3}$$

So you might say that the spring produces a force field for the mass to which it is attached; it defines a force and any point in space for the mass. Similarly, every charged particle produces an electric field. The electric field is defined for a particle of charge, q as

$$\vec{E} = k_e \frac{q}{r^2}\hat{r}. \tag{4}$$

Here the field for a point charge is written in spherical co-ordinates, but we might have just as easily written it in Cartesian coordinates as,

$$\vec{E} = k_e q \left(\frac{\hat{x}}{(x^2+y^2+z^2)^{3/2}} + \frac{\hat{y}}{(x^2+y^2+z^2)^{3/2}} + \frac{\hat{z}}{(x^2+y^2+z^2)^{3/2}} \right). \tag{5}$$

Clearly, the spherical coordinate version has the advantage of brevity, but we'll need this cartesian form form time to time for explicit calculations.

2.1 Coulombs Law

If we imagine for a moment that the magnetic field is zero everywhere, lets take a look at the form of the Lorentz force once more:

$$\vec{F} = q\vec{E} \tag{6}$$

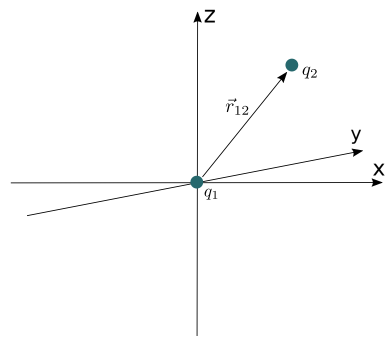
Now, imagine if the electric field was produced by a single particle of charge q' at the origin of our co-ordinate system. The force on our particle of interest would then take the form of the famous Coulomb's force law:

$$\vec{F} = k_e \frac{qq'}{r^2}\hat{r} \tag{7}$$

Note that this is the force on the particle at position $\vec{r}$ of charge q from the electric field produced by the particle at the origin of charge q'. More generally, the force between and two particles can be written as,

$$\vec{F} = k_e \frac{q_1 q_2}{{r_{12}}^2} \hat{r}_{12}. \tag{8}$$

Coulombs force law shows us that the electric field is the medium through which charged particles apply a force on one another. Importantly, we should check that this force satisfies Newtons third law.



Newtons third law states that every force has an equal and opposite force, or in our case this is the statement that the force from particle 1 on particle 2 needs to be equal in magnitude and opposite in direction to the force from particle 2 on particle 1. This is obviously the case because $\hat{r}_{12} = -\hat{r}_{21}$ and everything else is the same.

2.2 Fields from many charges and charge distributions

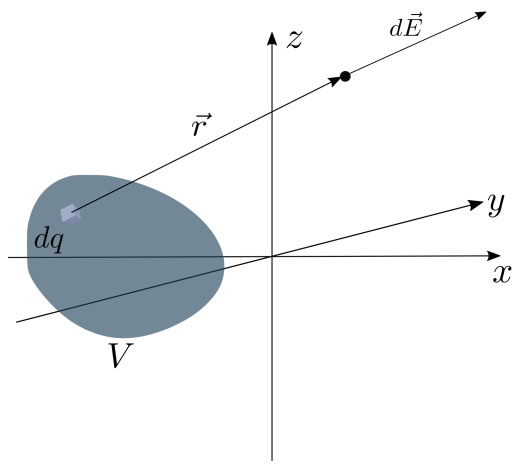
We will now consider the electric field produced by many charges. Electric field vectors, like force vectors, obey the law of superposition. This means that if we want to find the total electric field from a group of charged particles then we just need to *add* the contribution from each charge.

$$\vec{E}_{total}(\vec{r}) = \sum_i \vec{E}_i = \sum_i k_e \frac{q_i}{r_i^2} \hat{r}_i \tag{9}$$

More often then not, the typical situation is that the electric field at any given point has contributions from many millions of charged particles. This is the case if you have, for example, a balloon that has many electrons trapped on its surface after you rub it against your head or clothes. If this is the case, all is not lost! Though the charges are many, they are often evenly distributed with little space from charge to charge and we can use that to our advantage.

Instead of considering each individual charge we imagine the the total charge, Q is evenly smeared out across the surface, through the volume or along a line (which ever the circumstance dictates). If this is the case, instead of adding up the individual charges, we *integrate* the infinitisimal charges, dq:

$$\vec{E} = k_e \int_V \frac{\mathrm{d}q}{r^2}\hat{r} \tag{10}$$



In the case of constant charge distribution through a volume the integrand dq becomes ρdV where ρ is the charge distribution Q/V. In the case of a surface, we have instead $dq = \sigma dA$ and $dq = \lambda dl$ in the case of a line charge.

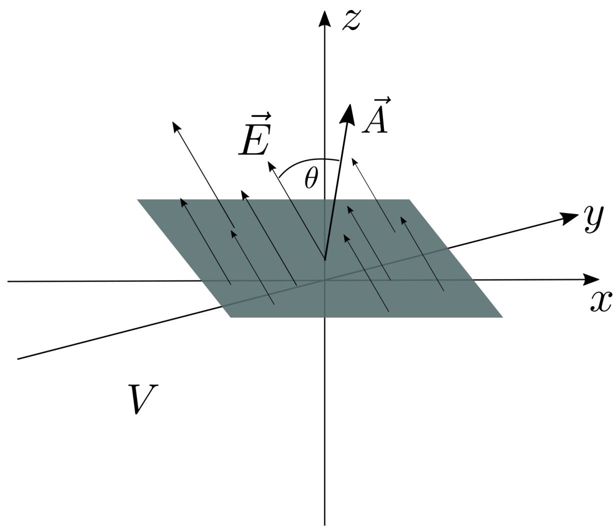
$$\vec{E} = k_e \int_V \frac{\rho(\vec{r})}{r^2}\mathrm{d}V\hat{r} \qquad \text{(Volume charge distribution)}$$
$$\vec{E} = k_e \int_S \frac{\sigma(\vec{r})}{r^2}\mathrm{d}A\hat{r} \qquad \text{(Surface charge distribution)}$$
$$\vec{E} = k_e \int_L \frac{\lambda(\vec{r})}{r^2}\mathrm{d}l\hat{r} \qquad \text{(Line charge distribution)}$$

3 Gauss's Law

Though in theory we've already seen a persciption for calculating the electric field from a charge distribution there are circumstances in which the problem is so high symmetric that we can use a trick named Gauss's Law to find the field without needing to solve *any* integral.

Gauss's Law requires that we define a few preliminary concepts. Gauss's Law concerns the possible surfaces in space that surround charge distribution so we need a way to talk about surfaces mathematically, and about electric fields near surfaces.

First we consider a flat plane of area A with constant electric field $\vec{E}$ everywhere on the surface. It is convenient for our purpose to introduce the idea of a surface vector $\vec{A}$ whose magnitude is the area of the plane and whose direction is perpendicular to the plane.



We then define the electric flux, Φ_e, through our surface A as the dot product of electric field with the surface vector.

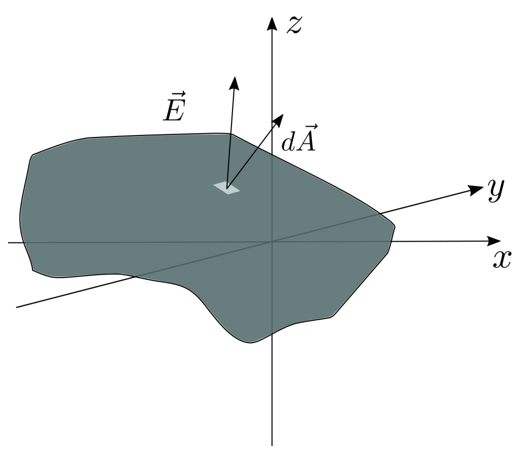
$$\Phi_e = \vec{E} \cdot \vec{A} = |\vec{E}||\vec{A}|cos(\theta) \tag{11}$$

For curved surfaces, we can approximate any surface as a sum of small flat surface elements ΔA_i and find the total flux through that surface to be the sum off all the fluxes through each small surface

$$\Phi_e = \sum_i \vec{E} \cdot \Delta\vec{A}_i \tag{12}$$

Or, should we take the limit where the surface elements become small we can integrate to find the total flux:

$$\Phi_e = \int_{surface} \vec{E} \cdot d\vec{A} \tag{13}$$



Gauss's Law is the remarkable statement that the total electric flux out of any closed surface is proportional to the charge contained inside the closed surface.

$$\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{q_{en}}{\epsilon_o} \tag{14}$$

The power of this statement is that if the charge distribution is sufficiently symmetric you can use this law to compute the electric field without doing *any* integrals! Here is the general methodology:

- Find a surface over which you can argue the electric field *must* be constant
- Make sure the electric field is either:
  1. Perpendicular to the surface
  2. Parallel to the surface or,
  3. Zero at the surface
- Use this fact to simplify the surface integral and factor our the electric field

These surfaces are called *Gaussian Surfaces* and they can make finding the electric field much easier is certain special circumstances.

4 Electric Potential and Potential Energy

You learned in mechanics that applying a force through some distance was called work and that work was a method for exchanging energy between two systems. For example, if I raise a bucket up off the the floor I have done work on the bucket by applying a force against gravity. The bucket will gain gravitational potential energy and I will loss some chemical potential energy (by burning calories to do the work).

With this in mind we consider the amount of work done moving a charged particle in an electric field and what type of energy might be exchanged. How much work is done by moving a particle of charge q through a path s under the influence of a electric field $\vec{E}$? At each point along the path the particle is moved a distance $d\vec{s}$ against the force $\vec{F} = q\vec{E}$ so, because we must supply the force opposite this and then add up all of the small $d\vec{s}$ contributions we find the total work done to be,

$$W = \Delta U_{potential} = -\int_{path} q\vec{E} \cdot d\vec{s}. \tag{15}$$

As stated above, the work done is also the potential energy gained by the particle. Note that the electric potential energy is a property of both the electric field and the particle together, but we'd like to define a type of potential energy that will describe the field regardless of the particles we subject to its presence. To that end, we can define the electric potential, $V = \Delta U/q$.

$$\Delta V = -\int_{path} \vec{E} \cdot d\vec{s} \tag{16}$$

Importantly, the potential is defined without respect to any test charge we might consider. We may also find the electric field if the electric potential is known. Consider the infinitismal path for equation 16:

$$dV = -\vec{E} \cdot d\vec{s} \Rightarrow \frac{dV}{dx} = -E_x \tag{17}$$

Generally you may find any component of the electric field by taking the negative of the derivative with respect that that co-ordinate.

There are a few very common scenerios that come up again and again in physical situations involving the potential. The first is the potential in regions of constant electric field. This is the case when we're between two large charged sheets of opposite charge. In this case, if we consider travel along the direction of the field lines:

$$\Delta V = -Ed \tag{18}$$

This is true, in fact in general as long as we consider the distance travelled, d, to be the distance travelled in the direction of the field lines.

Another common scenario is that of the potential of point particles. In this case we can define the potential far far away from the point charge to be zero and find the potential by integrating a particle path from out at infinity.

$$V = \frac{k_e q}{r} \tag{19}$$

Of course, from this we can find the potential energy of a particle pair to be

$$U = k_e \frac{q_1 q_2}{r_{12}} \tag{20}$$

4.1 Conservative Fields

An important feature of the electric potential is that the potential difference between two point does not depend on the path taken between those two points. Because of this, we define a general method to find the potental anywhere in space by integrating the electric field from infinity and defining the potential at infinity to be zero.

The property that the potential and hence the potential energy does not depend on the path of the work is called conservation. This is because any path taken will conserve energy. We say that the electric force is conservative and that the electric field is a conservative field.

4.2 Superposition

The electric potential from multiple sources is just the sum of the potential from each individual source. This can be seen as a consequence of conservation and superposition of the electric field. This means that to potential from many sources is simply,

$$V = \sum_i k_e \frac{q_i}{r_i}, \tag{21}$$

and of course we can extend this to distributions as we did for electric fields:

$$V = k_e \int \frac{dq}{r} \tag{22}$$

5 Conductors and Insulators

Thus far, we haven't really discussed the typically physical situation in which we find charged particles. At the most basic level we will consider only two microscopic charged particles: the electron and the proton. The electron and proton have a charge of $-e$ and $+e$ where e is the so called fundamental charge of 1.60218×10^-19C. This is the smallest charge that can be found in nature so all other charged particles are made from proton or electrons and therefore have some multiple of this charge

Almost all matter is composed of electrons bound to protons in the form on atoms so it is worth investigating some of the common characteristics of materials through the lense of these fundamental particles.

Since protons form the nucleus of the atom they do not often move significantly. On the other hand, electrons can either be bound strongly to their nucleas through the electrostatic force to flow freely from nucleus to nucleus. These two opposing behavours decribe insulators and conductors respectively. The perfect insulator allows no flow of electrons and thus any charge is "glued" in place. The perfect conductor on the other hand allows free flow of any charges.

Allowing the free flow of electrons can have some surprising consequences:

- The electric field inside any conductor must be zero
- All excess charge must reside on the surface of conductors
- Any conductor must have a constant potential

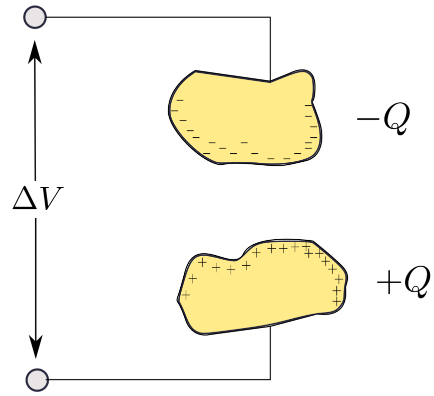
These three facts are all related to the fact that the only stable position for excess charges is pinned up against the wall of the conductor.

6 Capacitance and Dielectrics

A capacitor consists of two charged conductors that have equal and opposite charges. The capacitance of a capacitor is defined to be the charge of the capacitor Q divided by the potential difference between the conductors ΔV.

$$C = \frac{Q}{\Delta V} \tag{23}$$

The capacitor is so named because it can store electric potential energy. This is somewhat like a water resevour that can store gravitational potential energy. These capacitors are extremely important in electronic devices. The units of capacitance are farads $1F = 1C/1V$



6.1 Capacitors in Circuits

In a circuit capacitors can be single circuit elements in a larger circuit and there are similar rules for adding capacitors in parallel and series to find an equivalent capacitance. Namely, capacitors act opposite to resistros in that capacitance *adds* in parallel and adds reciprocally in series:

$$\frac{1}{C_{series}} = \sum_i \frac{1}{C_i} \tag{24}$$

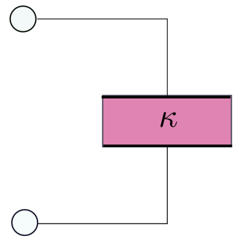
$$C_{parallel} = \sum_i C_i \tag{25}$$

As stated before, capacitors are used to store energy temporarily in circuits, but how much do they store? Consider trying to charge a capacitor from neutral.

$$\begin{aligned} dW &= \Delta V dq = \frac{q}{C} dq \Rightarrow \\ W &= \int_0^Q \frac{q}{C} dq \\ W &= \left. \frac{q^2}{2C} \right|_0^Q \\ W &= \frac{Q^2}{2C} = \frac{1}{2} Q \Delta V = \frac{1}{2} C (\Delta V)^2 \end{aligned} \tag{26}$$

Roughly speaking, the capacitor stores energy by holding difference charges apart from one another.

So far we've considered capacitors with only vacuum between them, but more often we choose to stick an insulating material between the capacitors. There is a practical reason for this as it helps keep the plates apart and there is no way to acheive a true vacuum between the conductors, but depending on the choice of material there can be an additional effect of increasing the capacitance.



All materials have a dielectric constant κ that is always greater than 1. The dielectric constant increases the capacitance of a capacitor via:

$$C = \kappa C_0 \tag{27}$$

Dielectric (materials with a dielectric constant less than infinity) increase the capacitance by keeping the charge constant (no current flows across) while they decrease the potential difference from node to node.

6.2 Microscopic theory of Dielectrics

The dielectric effect has a microscopic origin. Define the dipole to be two opposite charges seperated by a distance d. The dipole moment of a dipole is defined as,

$$p = dq. \tag{28}$$

Between the two plates of a capacitor the electric field is constant and therefore dipoles that are not in line with the field experience a torque,

$$\vec{\tau} = \vec{p} \times \vec{E}. \tag{29}$$

Futhermore, any misaligned dipoles incur and energy penalty of $U = -\vec{p} \cdot \vec{E}$. Therefore, if you fill the space between two capacitors with dipoles they will align with the field, but in doing so they will decrease the field. This reduces the potential difference and hence increases the capacitance.

7 Current and Resistance

One of the most important applications of electricity is in electronics and circuits. To discuss circuits we must define the current or electrical current. The current in a wire is defined to the the change in current with time, or,

$$I = \frac{dQ}{dt}. \tag{30}$$

The current density is said to be the full current through a wire divided by the wire cross sectional area.

$$J = I/A = nqv_d \tag{31}$$

Microscopically, the story reads as this: the current density is the carrier density times the charge per carrier time the drift velocity. The drift velocity is the ensemble averaged velocity.

7.1 Ohm's Law

Very often, we find that if we apply an electric field to a material and measure the current density, that the ratio of electric field to current density is a constant. We call this constant of proporionality the conductivity of the material and the relationship constitutes Ohm's Law.

$$\vec{J} = \sigma \vec{E} \tag{32}$$

Materials that follow this behaviour are said to be *ohmic* while those that violate it are said to be *nonohmic*. We may also integrate the current density to find the commonly used version of Ohm's Law $\Delta V = IR$. Where we have defined the resistant to be the recipocol of the conductivety time the volume. That is,

$$R = \rho \frac{l}{A}; \ \rho = \frac{1}{\sigma} \tag{33}$$

7.2 Microscopic Theory of Resistance

Drude theory predicts that the conductivity will be

$$\sigma = \frac{nq^2\tau}{m_e} \tag{34}$$

and the resistivity will be,

$$\rho = \frac{m_e}{nq^2\tau} \tag{35}$$

The resistance can also vary with the temperature. This is important when operating circuits across many different conditions. Your computor for instance must be able to continue operating across a broad range of temperatures to be useful to you. The temperature dependence of the resistivity is modelled empirically as

$$\rho = \rho_0 \left[1 + \alpha(T - T_0)\right] \tag{36}$$

where α is the temperature coefficient of resistivity. The usual case is that this coefficient is positive so the resistance grows with increased temperature, but this is not always the case.

7.3 Electrical Power

The power is the energy per unit time that is dissapated by the circuit. Over a resistor the interal energy is $q\Delta V$ so we find that:

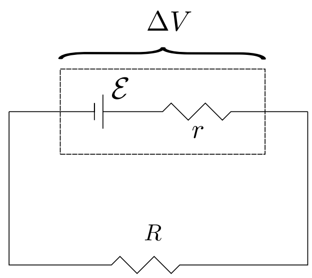
$$P = \frac{dU}{dt} = \frac{dQ\Delta V}{dt} = \Delta V I \tag{37}$$

This power over the resistor is said to be lost to Joules heating. This is how we heat our houses with electric baseboard heaters.

8 DC Circuits

All of the detailed properties of capacitors and resistors give us the capability to construct incredible devices. In this chapter we'll explore the properties of direct current circuits. Circuits are constructed from loops of conducting wire that are connected by various electrical componenets. The electrical components could be resistors, capacitors, potential sources and many other things. The unique electrical properties of each component determine the circuits overall behaviour.

In contrast so some ideal or hypothetical source of potential difference, our real life voltage sources, batteries, are not able to supply the same potential no matter what the circuit it is attached to, this is because batteries have an internal resistance that limits thier perceived potential output. We can model real batteries as an ideal potential source in series with a resistor.



The total potential difference is then the ideal potenial minus the potenial lost over the internal resistor:

$$\Delta V = \mathcal{E} - Ir \tag{38}$$

So, we can see that the potential output depends on the current and the internal resistance. An ideal potential source has no internal resistance.

8.1 Equivalent Resistance

An important technique in analysing circuits is make equivalent circuits. We've already seen that capacitors can be added to together to make an equivalent capacitor and the same is true for resistors. The rules for resistors is the opposite of the rules for capacitors and we can thing of it as follows.

The rule for resistors in series is that they must be summed to make an equivalent resistance.

$$R_{series} = R_1 + R_2 + R_3 \; ... \tag{39}$$

From the perspective of the electron this makes sense, instead of the facing one resistor it must now go through all of them one after the other so the resistance adds.

The rule for for resistors in parallel is that they are summed reciprocally to make an equivalent resistance.

$$\frac{1}{R_{parallel}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} ... \tag{40}$$

In effect this decreases the resistance overall and again from the perspective of the electron we can see why this would be so. We many resistors in parallel there are many more options to choose from. This is similar to adding area to a piece of metal.

Together both of these laws can be explained by the form of the resistance:

$$R = \rho \frac{l}{A} \tag{41}$$

8.2 Kirchoff's Rules

Kirchoff's Laws are convenient set of rules for analysing circuits. There are are two rules: The junction rule and the loop rule. The junction rule thats that the sum off all currents at and node must be equal to zero.

$$\sum_{\text{Node}} I = 0 \tag{42}$$

The junction rule is true because otherwise charge would pile on the nodes. The loop rules states that the sum of potential differences around any closed loop in a circuit must be zero.

$$\sum_{\text{Loop}} \Delta V = 0 \tag{43}$$

The loop rule is a restatement of the conservation of energy. It is important when using Kirchoff's rules that all current directions need to be defined *before* calculation. It does not matter what direction you choose for your currents, Kirchoffs laws will tell you the direction with respect to your convention when you solve the problem.

8.3 RC Circuits

An RC circuit is formed from a single loop with a capacitor, resistor and voltage source in series. Kirchoff's Law states that

$$\mathcal{E} - \frac{Q(t)}{C} - IR = 0 \tag{44}$$

We can solve this differential equation in time to find the charge as a function of time after the source as been connected:

$$Q(t) = Q_{total} \left[1 - e^{-t/RC}\right] \tag{45}$$

or, disconnected:

$$Q(t) = Q_{total} e^{-t/RC} \tag{46}$$

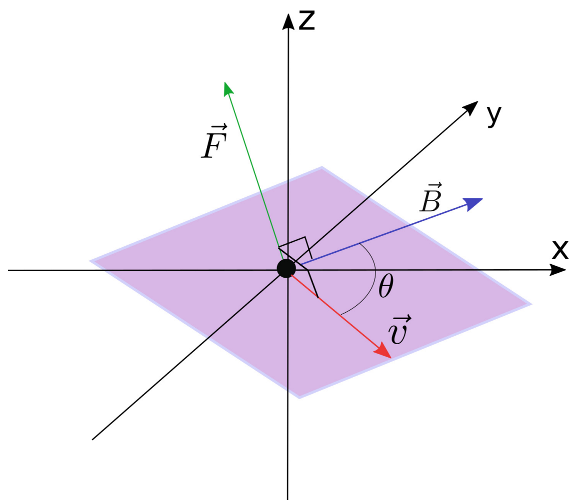
9 Magnetic Fields

The magnetic field is our second field of study. Before considering the sources of magnetic field we'll consider the dynamics of a particle that is under the influence of a constant magnetic field. Recall that from the Lorentz force law a particle under the influence of a magnetic field only has the following form:

$$\vec{F} = q(\vec{v} \times \vec{B}). \tag{47}$$

The cross product in the force can be off putting upon first glance so lets investigate some implication of this form to get in intuition for the magnetic effect. First, if the velocity of a particle is parallel to the magnetic field then there is no effect. At the other extreme, if the magnetic field is perpendicular to the velocity of the particle the magnitude of the force is simply $q|v||B|$ and we can find the direction using the right hand rule.

In general, the magnitude of the force will be $q|v||B|sin(\theta)$ where θ is the angle between the magnetic field and the particle velocity.



Note that force applied by the magnetic field is always perpendicular to the velocity of particle. This is not the first time we've seen a force of this nature. Remember when you studied the uniform circular motion that an accelaration or force perpendicular to the direction of travel lead to circular trajectories. This is the case once again, charged particles in constand magnetic field travel in circular trajectories.

Remembering the form of the centripedal force,

$$F_b = q|\vec{v}||\vec{B}| = \frac{m|\vec{v}|^2}{r}, \tag{48}$$

we can solve for the radius of the trajector,

$$r = \frac{m|\vec{v}|}{q|\vec{B}|}, \tag{49}$$

or the angular frequency,

$$\omega = \frac{q|\vec{B}|}{m}. \tag{50}$$

This frequency is called the *cyclotron frequency.*

Another interesting case of the magnetic force is a current carrying wire in an external field. Note the the $q\vec{v}$ term of the Lorentz force can be integrated over the entire wire to give a total force,

$$\vec{F}_b = I\vec{L} \times \vec{B}. \tag{51}$$

Where $\vec{L}$ is a vector along the wire with magnatude of the length of the wire exposed to the magnetic field. Of course this statement is true for an infinitismal wire as well so may integrate this affect over a wire path to give a totol force:

$$\vec{F}_B = I \int d\vec{s} \times \vec{B}. \tag{52}$$

One of the most critical examples of this effect is that of a current loop in an external field. This reason this is so important is that this is one of the main ingredients in making an electric motor. Here is why, if we consider a current loop (square for simplicity) in a constant field we can derive an expression for the torque as,

$$\vec{\tau} = I\vec{A} \times \vec{B}. \tag{53}$$

Here, the vector $\vec{A}$ is the vector normal to the loop with magnitude equal to the area of the loop. This means that if you apply a magnetic field in the plane of the loop you can push it upright. As the loop flips over if you apply an opposing magnetic field you can keep the loop rotating and thus create a motor. In fact we'll see later that this process can be run in reverse to make a generator, but we'll require Faraday's Law to know why.

We may further define the magnetic dipole moment in analogy with the electric dipole moment and show that a magnetic dipole can also store energy just as an electric dipole can.

$$\vec{\mu} = I\vec{A} \tag{54}$$

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

$$U = -\vec{\mu} \cdot \vec{B} \tag{55}$$

The energy stored in these magnetic moments underlies the physics of a vast number of materials as well as the physics of transformers and inductors.

10 Sources of the Magnetic Field

Where does the magnetic field come from? We'll just as stationary charged particles were affected by the electric field and so too gave rise to the electric field so it is with moving charged particles and creation of magnetic fields.

10.1 The Biot-Savart Law

In case of a general current path the form of the magnetic field is given by the *Biot-Savart* law.

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \hat{r}}{r^2} \tag{56}$$

The Biot-Savart law needs to be treated with special care when carrying out any calculations as the integrand is in fact a vector. This means that we're getting three equations for the price of one so to speak. The most salient feature at this point is the fact the the magnetic field is perpendicular to both the wire element and the unit position vector. The means that magnetic field lines tend to curl around current sources. In practice we don't often solve this equation, but instead rely on symmetry and Amperes law to find the magnetic field.

There are, though, some interesting results that can be derived from the Biot-Savart law such as the force between two parallel current wires:

$$\frac{F_B}{l} = \frac{\mu_0 I_1 I_2}{2\pi a} \tag{57}$$

In fact this very equation is used to define the unit of current, the Ampere.

10.2 Amperes Law

Ampere's law is yet another trick for calculating fields when the situation is highly symmetric, mush like Gauss law of electricity. Ampere's law states that the line integral of $\vec{B}$ around any closed path is equal to $\mu_0 I$ where I is the current enclosed by the loop.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I \tag{58}$$

Once again, if we choose a loop for which we know a priori that the magnetic field will be constant (by some symmetry argument) then we can find the magnetude of the field by solving Amperes Law. The technique of constructing an Amperian loop is much the same as making Gaussian surfaces.

One important application of Ampere's Law is in finding the magnetic field from a solenoid, which ends up being

$$B = \mu_0 \frac{N}{l} I = \mu_0 n I, \tag{59}$$

inside the loop, and exactly zero outside the loop.

10.3 Gauss's Law of Magnetism

There is an additional law of magnetism called Gauss's Law of magnetism. Formulating the magnetic flux exactly as we did the electric flux, Gauss's law simply states the the magnetic flux over any closed surface is exactly zero.

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{60}$$

11 Faradays Law

Faraday's Law is the underpinning of all modern electrical power generation. Whether nuclear, hydroelectric, or fuel burning all power plants eventually harness some mechanical energy to spin magnets around conducting coils which induces a current in those coils.

Precisely, Faraday's Law of induction states that you can produce an electromotive force in a conducting coil by changing the magnetic flux through that coil.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{61}$$

If, we consider the form of the magnetic flux, we see that there are a variety of ways to change the magnetic flux in time.

- change the magnetic field strength
- change the area of the current
- change the angle between the loop and the field

11.1 Lenz Law

Consider what is happening inside the conductor in the case of Faraday's Law. The EMF induced in the coil will be associated with a current, but as we know the current will also produce a magnetic field. In fact there the current produced in the coil will always be produced in such as a way that the magnetic field is makes will oppose try and restore the magnetic flux to its original form. This is the statement of Lenz's Law. Lenz's law is a good way to check if your calculations make sense.

So far we've discussed Faraday's Law in the sense that is produces an electromotive force inside of a wire. Of course we know that the magnetic field itself cannot be responsible for moving particles along the conducting loop, which makes us suspect that it is infact an electric field the drives the charges around the loop and in fact this is true. Faraday's law states that in general,

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\Phi_B}{dt} \tag{62}$$

12 Inductance

Just as we saw that a capacitor could store energy from the electric field it is also possible to store energy in the magnetic field using inductors. If we consider a circuit with an EMF source, a resistor and a switch, think about what happens when we close the switch. Current will start to flow and the current loop will produce a magnetic field inside the loop. Faradays law says that there will be an opposing EMF to oppose the change in magnetic flux. This *back*-EMF will be proportional to the change in current with respect to time and we call the constant of proportionality the inductance of the circuit

$$\mathcal{E} = -L\frac{d\Phi_B}{dt} \tag{63}$$

The inductance is measured in Henrys (1H = 1V s/A). For any loop the inductance has the simple form $L = N\Phi_B/I$, where N is the number of turns in the loop. If we recall the form of the resistance ($V = RI$) we can make a parallel between the inductance and the resistance: where the resistance is the opposition to current in a circuit, the inductance in the opposition to the *change* in current in a circuit.

12.1 RL Circuits

An RL circuit is a circuit composed of a single loop containing a resistor, inductor and voltage source. Solving Kirchoff's loop rule we find that the circuit must obey the following a differential equation and by solving this differential equation we find that the form of the current is exponential.

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \tag{64}$$

Specifically, the current as a function of time after the voltage is applied must be,

$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right), \tag{65}$$

where the time constant τ is the equal to R/L. Once the circuit reaches its peak current we can also solve for the current once the voltage source is turned off the current decays back to zero. This is simply an exponential decay with the same rate constant τ.

As stated previously, inductors can store energy in the magnetic field much as capacitors can store energy in the electric field. To see how this is true consider the work of "charging" an inductor to current I. We know that by multiplying Kirchoff's loop of an RL circuit by the current that power can be described as,

$$I\mathcal{E} = I^2R + IL\frac{dI}{dt}, \tag{66}$$

sense I^2R is the power dissapated by Joules heating at the resistor the second term must describe the power charging the inductor.

$$U = \int dU = \int_0^I LIdI = \frac{1}{2}LI^2 \tag{67}$$

12.2 Mutual Inductance

Consider the case of two coil that as coaxial and one lays above the other. If a current is passed through one coil, the magnetic field produced by the coil will change the magnetic flux through the second coil and this will induce a current in the second coil. This effect is called *Mutual inductance*. The mutual inductance of coil 1 on coil 2 is defined to be

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1}, \tag{68}$$

and importantly, the mutual inductane of coil 2 on coil 1 is the same. We refer to both simply as the mutual inductance M. Using Faraday's Law we can calculate the EMF induced in the opposite coil as,

$$\mathcal{E}_2 = -M \frac{dI_1}{dt} \tag{69}$$

This mutual inductance effect is how transformers work as well as induction cooking and wireless charging.